Cohomological Restrictions on Kähler groups

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Abstract

Let X be a compact Kähler manifold with fundamental group $\pi_1(X)$. After introducing the notion of higher Albanese genera g_k, the work establishes lower bounds on the number of the relations of $\pi_1(X)$ in terms of the number of the generators, the irregularity, the Albanese dimension, g_k and etc. The argument relates the cup product maps in the cohomologies of X and $\pi_1(X)$. It derives some lower bounds on the ranks of these cup products and applies Hopf's Theorem, describing $H_2(\pi_1(X), \mathbb{Z})$. The same techniques provide lower bounds on the Betti numbers of X and $\pi_1(X)$ within the range of the Albanese dimension.

1 Statement of the results

The abstract groups G which are isomorphic to the fundamental group $\pi_1(X)$ of a compact Kähler manifold X are briefly referred to as Kähler groups. These are always finitely presented.

The compact complex torus $Alb(X) = H^{1,0}(X)^*/H_1(X,\mathbb{Z})_{free}$ is called an Albanese variety of the compact Kähler manifold X. The Albanese map $alb_X : X \to Alb(X)$, $alb_X(x)(\omega) := \int_{x_0}^{x} \omega$ for $\omega \in H^{1,0}(X)$ is defined up to a translation, depending on the choice of a base point $x_0 \in X$. The Albanese dimension of X is $a = a(X) := \dim_{\mathbb{C}} alb_X(X)$.

The compact Kähler manifold Y is said to be Albanese general if $h^{1,0}(Y) > \dim_{\mathbb{C}} Y = a(Y)$. A surjective holomorphic map $f_k : X \to Y_k$ of a compact Kähler manifold X onto an Albanese general manifold Y_k of $\dim_{\mathbb{C}} Y_k = k$ is called an Albanese general k-fibration. It induces a complex linear embedding $f_k^* : H^{1,0}(Y_k) \to H^{1,0}(X)$ of the holomorphic $(1,0)$-forms, so that $h^{1,0}(Y_k)$ is bounded above by $h^{1,0}(X)$. The maximal $h^{1,0}(Y_k)$ for Albanese general k-fibrations $f_k : X \to Y_k$ is called k-th Albanese genus of X and denoted by $g_k = g_k(X)$.

The aim of the present note is to establish the following estimates:

Proposition 1 *Let X be a compact Kähler manifold whose fundamental group admits a finite presentation $\pi_1(X) = F/R$ where $F := \langle x_1, \ldots, x_s \rangle$ is a free group on s generators, $R_o := \langle y_1, \ldots, y_r \rangle$ is the subgroup of F, generated by the relations $y_1, \ldots, y_r \in F$ and R is the normal subgroup of F, generated by R_o. Suppose that the subgroup $K := (R_o \cap [F,R])/[R_o, R_o]$ of the abelianization $abR_o := R_o/[R_o, R_o] \simeq \mathbb{Z}^r$ is of $rkK = k$, $h^{1,0} := \dim_{\mathbb{C}} H^{1,0}(X)$ is the irregularity of X, a is the Albanese dimension and g_k, $1 \leq k \leq n$ are the Albanese genera of X. Then*

(i) $r \geq s + k$ *for* $h^{1,0} = 0$, $a = 0$;

(ii) $r \geq s + k - 2h^{1,0} + 1$ *for* $h^{1,0} \geq 1$, $a = 1$;

(iii) $r \geq s + k - 2h^{1,0} + \max(a(a-1), g_k(g_k - 1) \mid 2 \leq k \leq a) + \max\left(\frac{a(a-1)}{2}, 2a-1, g_k - 1 \mid 2 \leq k \leq a\right)$ *for* $h^{1,0} \geq g_1 \geq 2$, $a \geq 2$;

(iv) $r \geq s + k - 2h^{1,0} + \max(4h^{1,0} - 6, a(a-1), g_k(g_k-1) \mid 2 \leq k \leq a) +$

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$\max\left(2h^{1,0}-1, \frac{a(a-1)}{2}, g_k - 1 \;\mid\; 2 \le k \le a\right)$ *for* $h^{1,0} \geq 2,\ a \geq 2,\ g_1 = 0.$

A Kähler group $\pi_1(X)$ admits various finite presentations and there is no general algorithm for deciding whether two presentations determine isomorphic groups. The aforementioned Proposition 1 is not expected to perform Kähler tests on abstract finitely presented groups. It rather studies the influence of some cohomological properties of compact Kähler manifolds X on their fundamental groups $\pi_1(X)$. Part of the techniques from the proof of Proposition 1 provide also the following

Proposition 2 *Let X be a compact Kähler manifold with positive irregularity $h^{1,0}$, Albanese dimension $1 \leq a \leq n = \dim_{\mathbb{C}} X$ and Albanese genera g_k, $1 \leq k \leq n$. Then the Betti numbers $b_m(\pi_1(X)) := \dim_{\mathbb{C}} H^m(\pi_1(X), \mathbb{C})$ and $b_m(X) := \dim_{\mathbb{C}} H^m(X, \mathbb{C})$ are bounded below as follows :*

$$b_{2i}(\pi_1(X)) \geq 2\sum_{j=0}^{i-1} \mu^{j,2i-j} + \mu^{i,i}, \quad b_{2i+1}(\pi_1(X)) \geq 2\sum_{j=0}^{i} \mu^{j,2i+1-j} \quad \text{for } 3 \leq 2i, 2i+1 \leq a,$$

$$b_{2i}(X) \geq 2\sum_{j=0}^{i-1} \mu^{j,2i-j} + \mu^{i,i}, \quad b_{2i+1}(X) \geq 2\sum_{j=0}^{i} \mu^{j,2i+1-j} \quad \text{for } 3 \leq 2i, 2i+1 \leq a,$$

$$b_{2i}(X) \geq 2\sum_{j=0}^{i-1} \mu^{n-2i+j,n-j} + \mu^{n-i,n-i}, \quad b_{2i+1}(X) \geq 2\sum_{j=0}^{i} \mu^{n-2i-1+j,n-j} \quad \text{for } 2n-a \leq 2i, 2i+1 \leq 2n-3,$$

where

$$\mu^{i,j} := \max\left(\binom{a}{i+j}, \binom{g_k - i}{j}, \delta^0_{g_1} \dots \delta^0_{g_{i+j-1}}[(i+j)(h^{1,0}-i-j)+1] \;\mid\; g_k > 0, i+j \leq k \leq a \right)$$

for $i \leq j$ and $\delta^0_{g_s}$ standing for Kronecker's delta.

The next section specifies the cases in which the bounds from Proposition 1 are stronger than the already known results. Section 3 collects some properties of Albanese dimension and Albanese genera, necessary for deriving lower bounds on the ranks of cup products in $H^*(X, \mathbb{C})$. Section 4 relates cup products in group cohomologies $H^*(\pi_1(X), \mathbb{C})$ with the corresponding cup products in de Rham cohomologies $H^*(X, \mathbb{C})$. Section 5 justifies that $\mu^{i,m-i}$ from Proposition 2 are lower bounds on the ranks of the cup products $\zeta_X^{i,m-i} : \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^{m-i} H^{0,1}(X) \to H^m(X, \mathbb{C})$. The last section 6 recalls Hopf's Theorem on the second homologies of a group and concludes the proofs of Propositions 1 and 2.

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2 Comparison with previous related works

Prior to Proposition 1 are known the following estimates among the number of the generators and relations of a Kähler group.

Theorem 3 (Green and Lazarsfeld [7]) *Let X be a compact Kähler manifold whose fundamental group $\pi_1(X)$ admits a presentation with s generators and r relations.*

(i) If the Albanese genus $g_1 = 0$ then $r \geq s - 3$.

(ii) If the Albanese dimension $a \geq 2$ then $r \geq s - 1$.

For $s-r \geq 2$, Green and Lazarsfeld show that the entire character variety $\widehat{\pi_1(X)} = Hom(\pi_1(X), \mathbb{C}^*) \simeq H^1(X,\mathbb{C})$ of $\pi_1(X)$ is contained in the special locus $S^1(X) := \{L \in Pic^o(X) | H^1(X,L) \neq 0\}$ of the topologically trivial line bundles on X, parametrized by $Pic^o(X) \subset H^1(X, \mathcal{O}_X^*)$. Then there is a surjective holomorphic map $X \rightarrow C$ onto a curve C of genus $\geq \frac{s-r}{2}$ and the Albanese image of X is a curve. That violates the assumption of the second part. The first part is contradicted by $s-r \geq 4$, as far as $g_1 = 0$ signifies the nonexistence of surjective holomorphic maps $X \rightarrow C$ onto curves C of genus ≥ 2.

Theorem 4 (Amorós [1]) *Let X be a compact Kähler manifold with $g_1 = 0$, whose fundamental group $\pi_1(X)$ admits a presentation with s generators and r relations. Then*
(i) $r \geq s$ for $h^{1,0} = 0$;
(ii) $r \geq s-1$ for $h^{1,0} = 2$;
(iii) $r \geq s + 4h^{1,0} - 7$ for $h^{1,0} \geq 2$.

Let $\zeta_X^2 : \wedge^2 H^1(X,\mathbb{C}) \rightarrow H^2(X,\mathbb{C})$ be the cup product in de Rham cohomologies. Making use of Sullivan's 1-minimal models, Amorós identifies $Ker\zeta_X^2$ with $(\pi_1(X)_2/\pi_1(X)_3) \otimes \mathbb{R}$ where $\pi_1(X)_1 := \pi_1(X)$, $\pi_1(X)_{i+1} := [\pi_1(X)_i, \pi_1(X)]$ are the components of the lower central series of the fundamental group $\pi_1(X)$.

For an arbitrary group G, let $J_G := \{\sum_g r_g g | r_g \in \mathbb{R}, \sum_g r_g = 0\}$ be the augmentation ideal of the group ring $\mathbb{R}[G]$. It is well known (cf.[10]) that $H_1(G,\mathbb{Z}) \simeq G/[G,G] \simeq J_G/J_G^2$. Amorós shows in [1] that the $\mathbb{R}$-linear map $\Delta_1 : \oplus_{j=1}^r \mathbb{R} y_j \rightarrow J_F/J_F^2$, $\Delta_1(y_j) = y_j - 1 + J_F^2$ has $Coker\Delta_1 = J_{\pi_1(X)}/J_{\pi_1(X)}^2$. In particular, $\dim_{\mathbb{R}} Ker\Delta_1 = r - s + 2h^{1,0}$. Further, the induced map $\Delta_2 : Ker\Delta_1 \rightarrow \wedge^2 \left(J_{\pi_1(X)}/J_{\pi_1(X)}^2 \right)$ is proved to have $Coker\Delta_2 \simeq (\pi_1(X)_2/\pi_1(X)_3) \otimes \mathbb{R}$. Consequently, $\dim_{\mathbb{R}} Ker\zeta_X^2 = \dim_{\mathbb{R}} (\pi_1(X)_2/\pi_1(X)_3) \otimes \mathbb{R} = \begin{pmatrix} 2h^{1,0} \\ 2 \end{pmatrix} - \dim_{\mathbb{R}} Ker\Delta_1 + \dim_{\mathbb{R}} Ker\Delta_2 \geq \begin{pmatrix} 2h^{1,0} \\ 2 \end{pmatrix} - r + s - 2h^{1,0}$ or $rk\zeta_X^2 \leq r - s + 2h^{1,0}$.

On the other hand, Amorós makes use of the following lower bounds on the ranks of the cup products:

Lemma 5 *If X is a compact Kähler manifold with $g_1 = 0$ then*
(i) $rk[\zeta_X^{2,0} : \wedge^2 H^{1,0}(X) \rightarrow H^2(X,\mathbb{C})] \geq 2h^{1,0} - 3$;
(ii) $rk[\zeta_X^{1,1} : H^{1,0}(X) \otimes_{\mathbb{C}} H^{0,1}(X) \rightarrow H^2(X,\mathbb{C})] \geq 2h^{1,0} - 1$.

The estimate *(i)* is derived from the transversality of the cone $\mathcal{C}^{2,0} := \{\omega_1 \wedge \omega_2 | \omega_1, \omega_2 \in H^{1,0}(X)\}$ to the kernel of $\zeta_X^{2,0}$, i.e., $\mathcal{C}^{2,0} \cap Ker\zeta_X^{2,0} = \{0\}$ (cf. also [2].) The inequality *(ii)* is a consequence of a theorem of Remmert and Van de Ven from [9]. It asserts that a holomorphic map $\tau : A_1 \times A_2 \rightarrow B$ of projective algebraic manifolds A_1, A_2 in a complex space B has $rk_{\mathbb{C}}\tau \geq \dim_{\mathbb{C}} A_1 + \dim_{\mathbb{C}} A_2$, provided $b_2(A_1) = b_2(A_2) = 1$, $b_1(A_i) = 0$ for some $1 \leq i \leq 2$ and τ does not factor through a canonical projection $\Pi_i : A_1 \times A_2 \rightarrow A_i$. This is applied to the projectivization $\mathbf{P}(\zeta_X^{1,1}) : \mathbf{P}(H^{1,0}(X)) \times \mathbf{P}(H^{0,1}(X)) \rightarrow \mathbf{P}(H^2(X,\mathbb{C}))$ of the bilinear map $\zeta_X^{1,1}$ with trivial kernel.

Obviously, Theorem 4 implies Theorem 3*(i)*. One checks straightforward that the inequalities, given by *(i)*, *(ii)* with $h^{1,0} = 1$ and *(iv)* from Proposition 1 are stronger than the corresponding estimates from Theorem 4. In the case of $a \geq 2$, $h^{1,0} \geq g_1 \geq 2$ with $h^{1,0}$ comparatively large with respect to a and g_k, $2 \leq k \leq a$, the bound from Proposition 1 *(iii)* may happen to be weaker than the one from Theorem 3 *(ii)*.

3 Preliminaries on Albanese dimension and Albanese genera

Some of the bounds on the ranks of cup products, proved in section 5, require the characterization the Albanese dimension in terms of cup products.

Proposition 6 (Catanese [3],[5]) *The Albanese dimension $a := \dim_{\mathbb{C}} alb_X(X)$ of a compact Kähler manifold X is the greatest integer with $Im[\zeta_X^{a,a} : \wedge^a H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^a H^{0,1}(X) \rightarrow H^{2a}(X,\mathbb{C})] \neq 0$ or, equivalently, the greatest integer with $Im[\zeta_X^{a,0} : \wedge^a H^{1,0}(X) \rightarrow H^a(X,\mathbb{C})] \neq 0$.*

The following trivial observations were probably a starting point for Catanese's Theorem 8:

Lemma 7 *(i) For an arbitrary Albanese general compact Kähler manifold Y of $\dim_{\mathbb{C}} Y = k$, the cup product $\zeta_Y^{k,0} : \wedge^k H^{1,0}(Y) \to H^k(Y, \mathbb{C})$ is injective and the cup product $\zeta_Y^{k+1,0} : \wedge^{k+1,0} H^{1,0}(Y) \to H^{k+1}(Y, \mathbb{C})$ is identically zero.*

(ii) If $f_k : X \to Y_k$ is an Albanese general k-fibration and $U_k := f_k^* H^{1,0}(Y_k)$ then $Ker[\zeta_X^{k,0} : \wedge^k U_k \to H^k(X, \mathbb{C})] = 0$ and $Im[\zeta_X^{k+1,0} : \wedge^{k+1} U_k \to H^{k+1}(X, \mathbb{C})] = 0$. Such subspaces $U_k \subset H^{1,0}(X)$ are called strict k-wedges.

Proof: *(i)* According to Proposition 6, there exist $\omega_1, \ldots, \omega_k \in H^{1,0}(Y)$ with $\zeta_Y^{k,0}(\omega_1 \wedge \ldots \wedge \omega_k) \neq 0$. Let $\{W^{(\alpha)}\}_{\alpha \in A}$ be a coordinate covering of Y. For any $\omega \in H^{1,0}(Y)$ there exist local meromorphic functions $\mu_i^{(\alpha)} : W^{(\alpha)} \to \mathbb{C}\mathbf{P}^1$, such that $\omega|_{W^{(\alpha)}} = \sum_{i=1}^k \mu_i^{(\alpha)} \omega_i|_{W^{(\alpha)}}$. On the overlaps $W^{(\alpha)} \cap W^{(\beta)} \neq \emptyset$, one has $\sum_{i=1}^k \left(\mu_i^{(\alpha)} - \mu_i^{(\beta)} \right) \omega_i|_{W^{(\alpha)} \cap W^{(\beta)}} \equiv 0$ since ω and $\omega_1, \ldots, \omega_k$ are globally defined. The linear independence of $\omega_1, \ldots, \omega_k$ is inherited by their restrictions on the open subset $W^{(\alpha)} \cap W^{(\beta)}$ of Y and implies $\mu_i^{(\alpha)}|_{W^{(\alpha)} \cap W^{(\beta)}} = \mu_i^{(\beta)}|_{W^{(\alpha)} \cap W^{(\beta)}}$ for all $1 \leq i \leq k$. In other words, $\mu_i : Y \to \mathbb{C}\mathbf{P}^1$ are globally defined and $\omega \in H^{1,0}(Y)$ can be globally represented in the form $\omega = \sum_{i=1}^k \mu_i \omega_i$. Consequently, $\wedge^k H^{1,0}(Y)$ consists of $\mu \omega_1 \wedge \ldots \wedge \omega_k$ for some global meromorphic functions $\mu : Y \to \mathbb{C}\mathbf{P}^1$. The assumption $\zeta_Y^{k,0}(\mu \omega_1 \wedge \ldots \wedge \omega_k) = 0$ is equivalent to the existence of a $(k-1)$-form σ with $\mu \omega_1 \wedge \ldots \wedge \omega_k = d\sigma$. Then $0 = \int_X d(\sigma \wedge d\overline{\sigma}) = \int_X |\mu|^2 \omega_1 \wedge \ldots \wedge \omega_k \wedge \overline{\omega_1} \wedge \ldots \wedge \overline{\omega_k}$ forces the vanishing of μ almost everywhere on Y. According to the complex analyticity of the zero locus of μ, one concludes that $\mu \equiv 0$, i.e., $\zeta_Y^{k,0} : \wedge^k H^{1,0}(Y) \to H^k(Y, \mathbb{C})$ is injective. Clearly, $\zeta_Y^{k+1,0}(\wedge^{k+1} H^{1,0}(Y)) = 0$ due to $\dim_{\mathbb{C}} Y = k$.

(ii) The surjective holomorphic map $f_k : X \to Y_k$ induces an embedding $f_k^* : H^*(Y_k, \mathbb{C}) \to H^*(X, \mathbb{C})$, compatible with the cup products. More precisely,

$$0 = \zeta_X^{l,0} \left(\sum_{i=(i_1, \ldots, i_l)} f_k^*(\omega_{i_1}) \wedge \ldots \wedge f_k^*(\omega_{i_l}) \right) = \zeta_X^{l,0} f_k^* \left(\sum_{i=(i_1, \ldots, i_l)} \omega_{i_1} \wedge \ldots \wedge \omega_{i_l} \right) =$$

$$f_k^* \zeta_{Y_k}^{l,0} \left(\sum_{i=(i_1, \ldots, i_l)} \omega_{i_1} \wedge \ldots \wedge \omega_{i_l} \right)$$

is equivalent to $\zeta_{Y_k}^{l,0} \left(\sum_{i=(i_1, \ldots, i_l)} \omega_{i_1} \wedge \ldots \wedge \omega_{i_l} \right) = 0$ for any natural number l. Putting $l = k$ or $k+1$ and combining with *(i)*, one obtains *(ii)*, Q.E.D.

Here is the generalized Castelnuovo-deFranchis Theorem:

Theorem 8 (Catanese [5]) *Let X be a compact Kähler manifold. Then for any strict k-wedge $U_k \subset H^{1,0}(X)$ there exists an Albanese general k-fibration $f_k : X \to Y_k$ with $f_k^* H^{1,0}(Y_k) = U_k$, which is unique up to a biholomorphism of Y_k.*

Corollary 9 *The k-th Albanese genus g_k of a compact Kähler manifold X equals the maximum dimension of a strict k-wedge $U_k \subset H^{1,0}(X)$ of $\dim_{\mathbb{C}} U_k > k$.*

In order to formulate one more result of Catanese, used in section 5, let us say that $V_k \subset H^{1,0}(X)$ is a k-wedge if $\zeta_X^{k,0}(\wedge^k V_k) \neq 0$ and $\zeta_X^{k+1,0}(\wedge^{k+1} V_k) = 0$.

Lemma 10 (Catanese [5]) *If X is a compact Kähler manifold then any k-wedge $V_k \subset H^{1,0}(X)$ contains a strict l-wedge $U_l \subseteq V_k$ for some natural number $l \leq k$.*

4 Cup products in group and de Rham cohomologies

Let us choose a cell decomposition of X. Then construct an Eilenberg-MacLane space $Y = K(\pi_1(X), 1)$ by glueing cells of real dimension ≥ 3 to X, in order to annihilate the higher homotopy groups $\pi_i(X)$, $i \geq 2$. Put $c : X \to Y$ for the resulting classifying map and denote by $S(X)_\bullet$, $S(Y)_\bullet$ the corresponding singular chain complexes. The induced chain morphism $c_* : S(X)_\bullet \to S(Y)_\bullet$ is an isomorphism in degree ≤ 2 and injective in degree $i \geq 3$. If ∂_i^X, ∂_i^Y are the boundary maps $\partial_i^* : S(*)_i \to S(*)_{i-1}$ and $Z(*)_i := \{\xi \in S(*)_i | \partial_i^*(\xi) = 0\}$ are the abelian subgroups of the cycles, then $c_i : H_i(X,\mathbb{Z}) := Z(X)_i/\partial_{i+1}^X S(X)_{i+1} \to H_i(\pi_1(X),\mathbb{Z}) = H_i(Y,\mathbb{Z}) := Z(Y)_i/\partial_{i+1}^Y S(Y)_{i+1}$ are isomorphisms for $i = 0, 1$ and $c_2 : H_2(X,\mathbb{Z}) \to H_2(\pi_1(X),\mathbb{Z})$ is surjective. In general, the homomorphisms of abelian groups $c_i : H_i(X,\mathbb{Z}) \to H_i(\pi_1(X),\mathbb{Z})$ for $i \geq 3$ do not obey to any specific restrictions.

For any field F of $char F = 0$, acted trivially by $\pi_1(X)$, the Universal Coefficients Theorems

$$0 \to Ext^1_{\mathbb{Z}}(H_{m-1}(X,\mathbb{Z}), F) \to H^m(X,F) \to Hom_{\mathbb{Z}}(H_m(X,\mathbb{Z}), F) \to 0,$$

$$0 \to Ext^1_{\mathbb{Z}}(H_{m-1}(\pi_1(X),\mathbb{Z}), F) \to H^m(\pi_1(X),F) \to Hom_{\mathbb{Z}}(H_m(\pi_1(X),\mathbb{Z}), F) \to 0,$$

provide $H^m(\quad, F) \simeq F^{rkH_m(\ ,\mathbb{Z})}$, due to the divisibility of the $\mathbb{Z}$-module F. In particular, the $\mathbb{C}$-linear maps $c^i : H^i(\pi_1(X),\mathbb{C}) = H^i(Y,\mathbb{C}) \to H^i(X,\mathbb{C})$ are isomorphisms for $i = 0, 1$ and injective for $i = 2$. Making use of the Hodge decomposition $H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$ on the first cohomologies of a compact Kähler manifold X, one introduces $H^{k,l}(\pi_1(X)) := \left(c^1\right)^{-1} H^{k,l}(X)$ for $(k,l) = (1,0)$ or $(0,1)$. On one hand, there are cup products $\zeta^{i,j}_{\pi_1(X)} : \wedge^i H^{1,0}(\pi_1(X)) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(\pi_1(X)) \to H^{i+j}(\pi_1(X),\mathbb{C})$ of group cohomologies, defined as a composition of the direct product with the dual of a diagonal approximation (cf.[4]). On the other hand, one has cup products $\zeta^{i,j}_X : \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X,\mathbb{C})$ of de Rham cohomologies. Their images are related by the following

Lemma 11 *Let X be a compact Kähler manifold with fundamental group $\pi_1(X)$, $Y = K(\pi_1(X),1)$ be an Eilenberg-MacLane space and $c : X \to Y$ be a continuous classifying map. Then the cup products*

$$\zeta^{i,j}_{\pi_1(X)} : \wedge^i H^{1,0}(\pi_1(X)) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(\pi_1(X)) \to H^{i+j}(\pi_1(X),\mathbb{C}) \quad \textit{and}$$

$$\zeta^{i,j}_X : \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X,\mathbb{C})$$

have $c^{i+j} Im\zeta^{i,j}_{\pi_1(X)} = Im\zeta^{i,j}_X$. In particular, $rk\zeta^{i,j}_{\pi_1(X)} \geq rk\zeta^{i,j}_X$ for all $i, j \in \mathbb{N} \cup \{0\}$, $i + j \in \mathbb{N}$.

Proof: As far as $Y = K(\pi_1(X),1)$ and $c : X \to Y$ are unique up to homotopy, there is no loss of generality in assuming that Y is obtained from X by glueing cells of real dimension ≥ 3. Then $c : X \to Y$ and the chain morphism $c_* : S(X)_\bullet \to S(Y)_\bullet$ are inclusion maps and the dual cochain morphism $c^* : S(Y)^\bullet := Hom_{\mathbb{Z}}(S(Y)_\bullet, \mathbb{C}) \to S(X)^\bullet := Hom_{\mathbb{Z}}(S(X)_\bullet, \mathbb{C})$ is a surjective restriction from Y to X.

By induction on i one checks that $c^i \zeta^i_Y (\wedge^i S(Y)^1) = \zeta^i_X(\wedge^i S(X)^1)$ for the cup products $\zeta^i_* : \wedge^i S(*)^1 \to S(*)^i$ of cochains. The case of $i = 1$ is straightforward from the construction of Y. Any cell $\sigma_i \in S(X)_i$ is homotopy equivalent to a product of segments $[a_1, b_1] \times \ldots \times [a_i, b_i]$. Let us put $\sigma'_i := [a_1, b_1] \times \ldots \times [a_{i-1}, b_{i-1}]$, $\sigma''_i := [a_i, b_i]$ and choose some $u_1, \ldots, u_i \in S(Y)^1$. Representing the singular cochains on the manifold X by smooth differential forms, one has $\int_{\sigma'_i} c^{i-1} \zeta^{i-1}_{\pi_1(X)}(u_1 \wedge \ldots \wedge u_{i-1}) = \int_{\sigma'_i} \zeta^{i-1}_X (c^1(u_1) \wedge \ldots \wedge c^1(u_{i-1}))$, by the inductional hypothesis. Then $S(X)^1 = c^1 S(Y)^1$ and Fubini's Theorem provide

$$\int_{\sigma_i} \zeta^i_X (c^1(u_1) \wedge \ldots \wedge c^1(u_i)) = \int_{\sigma'_i} \zeta^{i-1}_X (c^1(u_1) \wedge \ldots \wedge c^1(u_{i-1})) \int_{\sigma''_i} c^1(u_i) =$$

$$\int_{\sigma'_i} c^{i-1} \zeta^{i-1}_{\pi_1(X)} (u_1 \wedge \ldots \wedge u_{i-1}) \int_{\sigma''_i} c^1(u_i) = \int_{\sigma_i} c^i \zeta^i_{\pi_1(X)} (u_1 \wedge \ldots \wedge u_{i-1} \wedge u_i)$$

under the choice of a diagonal approximation $\Delta : S(Y)_\bullet \to S(Y)_\bullet \otimes S(Y)_\bullet$, $\Delta([a_1,b_1]\times \ldots \times [a_m,b_m]) = \sum_{j=0}^{m}([a_1,b_1]\times \ldots \times [a_j,b_j]) \otimes ([a_{j+1},b_{j+1}]\times \ldots \times [a_m,b_m])$ on the singular chains of the Eilenberg-MacLane space $Y = K(\pi_1(X),1)$. As far as the cup products ζ_X^i, ζ_Y^i and the cochain maps $c^i : S(Y)^i \to S(X)^i$ are $\mathbb{C}$-linear, there follows $c^i\zeta_Y^i(\wedge^i S(Y)^1) = \zeta_X^i(\wedge^i S(X)^1)$ for all $i \in \mathbb{N}$.

The restriction $c^* : S(Y)^\bullet \to S(X)^\bullet$ is a morphism of cochain complexes, so that commutes with the coboundary maps $\delta_*^i : S(*)^i \to S(*)^{i+1}$, i.e., $\delta_X^i c^i = c^{i+1}\delta_Y^i$. In particular, the isomorphisms $c^i : S(Y)^i \to S(X)^i$ for $i = 1,2$ induce an isomorphism of the 1-cocycles $c^1 : Z(Y)^1 \to Z(X)^1$, where $Z(*)^1 := \{\xi \in S(*)^1 \ | \ \delta_*^1(\xi) = 0\}$. Representing the elements of $Z(X)^1$ by closed differential forms and making use of the complex structure J on X, one decomposes $Z(X)^1 = Z(X)^{1,0} \oplus Z(X)^{0,1}$ into a direct sum of $\pm\sqrt{-1}$-eigenspaces for the action of J. That allows to introduce $Z(Y)^{k,l} := (c^1)^{-1}Z(X)^{k,l}$ for $(k,l) = (1,0)$ or $(0,1)$ and to represent $Z(Y)^1 = Z(Y)^{1,0} \oplus Z(Y)^{0,1}$. Leibnitz' rule for the coboundary maps δ_X^{i+j} justifies the existence of natural cup products $\zeta_X^{i,j} : \wedge^i Z(X)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(X)^{0,1} \to Z(X)^{i+j}$. As far as $\wedge^{i+j}(c^1)^{-1} : \wedge^i Z(X)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(X)^{0,1} \to \wedge^i Z(Y)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(Y)^{0,1}$ are well defined isomorphisms, compatible with δ_X^*, δ_Y^*, one can introduce cup products $\zeta_Y^{i,j} : \wedge^i Z(Y)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(Y)^{0,1} \to Z(Y)^{i+j}$ with $c^{i+j}\zeta_Y^{i,j}(\wedge^i Z(Y)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(Y)^{0,1}) = \zeta_X^{i,j}(\wedge^i Z(X)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(X)^{0,1})$. On the other hand, the surjectiveness of the cochain morphism c^* implies that $c^{i+j-1}S(Y)^{i+j-1} = S(X)^{i+j-1}$, whereas $c^{i+j}\delta_Y^{i+j-1}S(Y)^{i+j-1} = \delta_X^{i+j-1}c^{i+j-1}S(Y)^{i+j-1} = \delta_X^{i+j-1}S(X)^{i+j-1}$.

Consequently,

$$c^{i+j}Im\zeta_Y^{i,j} = \frac{c^{i+j}\zeta_Y^{i,j}\left(\wedge^i Z(Y)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(Y)^{0,1}\right)}{c^{i+j}\zeta_Y^{i,j}\left(\wedge^i Z(Y)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(Y)^{0,1}\right) \cap c^{i+j}\delta_Y^{i+j}S(Y)^{i+j-1}} =$$

$$\frac{\zeta_X^{i,j}\left(\wedge^i Z(X)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(X)^{0,1}\right)}{\zeta_X^{i,j}\left(\wedge^i Z(X)^{1,0} \otimes_{\mathbb{C}} \wedge^j Z(X)^{0,1}\right) \cap \delta_X^{i+j-1}S(X)^{i+j-1}} = Im\zeta_X^{i,j}, \quad \text{Q.E.D.}$$

5 Estimates on cup products

The present section provides lower bounds on the rank of cup products of 1-forms on X. Clearly, $rk\zeta_X^{1,0} = rk\zeta_X^{0,1} = h^{1,0}$, as far as $\zeta_X^{i,j} = Id_{H^{i,j}(X)}$ for $(i,j) = (1,0)$ or $(0,1)$.

Lemma 12 *Let X be a compact Kähler manifold with Albanese dimension a. Then the cup products*

$$\zeta_X^{i,j} : \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X,\mathbb{C})$$

factor through the cup products $\zeta_X^{p,q}$ for all $0 \leq p \leq i$, $0 \leq q \leq j$. In particular, $\zeta_X^{i,j} \equiv 0$ for $i > a$ or $j > a$ and the Albanese genera $g_k = 0$ for all $k > a$.

Proof: Let us identify the cohomology classes on X with their de Rham representatives. Denote by $A^{r,s}$ the space of the C^∞-forms of type (r,s) and put $Z^{r,s} = \{\varphi \in A^{r,s} | d\varphi = 0\}$ for the subspace of the d-closed forms. Then $\wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) = (\wedge^i Z^{1,0} \otimes_{\mathbb{C}} \wedge^j Z^{0,1})/L$, $Im\zeta_X^{i,j} = (\wedge^i Z^{1,0} \otimes_{\mathbb{C}} \wedge^j Z^{0,1})/M$, $Im(\zeta_X^{p,q} \wedge Id_{\wedge^{i-p}H^{1,0}(X)\otimes_{\mathbb{C}}\wedge^{j-q}H^{0,1}(X)} = (\wedge^i Z^{1,0} \otimes_{\mathbb{C}} \wedge^j Z^{0,1})/N$ where

$$L := [(dA^{0,0} \cap A^{1,0}) \wedge (\wedge^{i-1}Z^{1,0})] \otimes_{\mathbb{C}} \wedge^j Z^{0,1} + \wedge^i Z^{1,0} \otimes_{\mathbb{C}} [(dA^{0,0} \cap A^{0,1}) \wedge (\wedge^{j-1}Z^{0,1})],$$

$$M := (dA^{i-1,j} + dA^{i,j-1}) \cap A^{i,j}, \quad N := [(dA^{p-1.q} + dA^{p,q-1}) \cap A^{p,q}] \wedge (\wedge^{i-p}Z^{1,0} \otimes_{\mathbb{C}} \wedge^{j-q}Z^{0,1}) +$$

$$Z^{p,q} \wedge \{[(dA^{0,0} \cap A^{1,0}) \wedge (\wedge^{i-p-1}Z^{1,0})] \otimes_{\mathbb{C}} \wedge^{j-q}Z^{0,1}\} + Z^{p,q} \wedge \{\wedge^{i-p}Z^{1,0} \otimes_{\mathbb{C}} [(dA^{0,0} \cap A^{0,1}) \wedge (\wedge^{i-q-1}Z^{0,1})]\}.$$

The existence of correctly defined $\mathbb{C}$-linear maps $\zeta_X^{i,j}$, $\zeta_X^{p,q} \wedge Id_{\wedge^{i-p}H^{1,0}(X)\otimes_{\mathbb{C}}\wedge^{j-q}H^{0,1}(X)}$ for $0 \leq p \leq i$, $0 \leq q \leq j$ is due to the inclusions $L = d(A^{0,0} \otimes_{\mathbb{C}} \wedge^{i-1}Z^{1,0} \otimes_{\mathbb{C}} \wedge^j Z^{0,1}) \cap A^{i,j} + d(\wedge^i Z^{1,0} \otimes_{\mathbb{C}} A^{0,0} \otimes_C \wedge^{j-1}Z^{0,1}) \cap A^{i,j} \subseteq M$, $L = (\wedge^p Z^{1,0} \otimes_{\mathbb{C}} \wedge^q Z^{0,1}) \wedge \{[(dA^{0,0} \cap A^{1,0}) \wedge (\wedge^{i-p-1}Z^{1,0})] \otimes_{\mathbb{C}} \wedge^{j-q}Z^{0,1}\} + (\wedge^p Z^{1,0} \otimes_{\mathbb{C}} \wedge^q Z^{0,1}) \wedge \{\wedge^{i-p}Z^{1,0} \otimes_{\mathbb{C}}$

$[(da^{0,0} \cap A^{0,1}) \wedge (\wedge^{j-q-1} Z^{0,1})] \subseteq N$. A necessary and sufficient condition for the factorization of $\zeta_X^{i,j}$ through $\zeta_X^{p,q} \wedge Id_{\wedge^{i-p} H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^{j-q} H^{0,1}(X)}$ is the inclusion $N = d(A^{p-1,q} \otimes_{\mathbb{C}} \wedge^{i-p} Z^{1,0} \otimes_{\mathbb{C}} \wedge^{j-q} Z^{0,1}) \cap A^{i,j} + d(A^{p,q-1} \otimes_{\mathbb{C}} \wedge^{i-p} Z^{1.0} \otimes_{\mathbb{C}} \wedge^{j-q} Z^{0,1}) \cap A^{i,j} + d(Z^{p,q} \otimes_{\mathbb{C}} A^{0,0} \otimes_{\mathbb{C}} \wedge^{i-p-1} Z^{1,0} \otimes_{\mathbb{C}} \wedge^{j-q} Z^{0,1} \cap A^{i,j} + d(Z^{p,q} \otimes_{\mathbb{C}} \wedge^{i-p} Z^{1,0} \otimes_{\mathbb{C}} A^{0,0} \otimes_{\mathbb{C}} \wedge^{j-q-1} Z^{0,1}) \cap A^{i,j} \subseteq M$. In particular, $\zeta_X^{i,j}$ with $i > a$ factor through $\zeta_X^{a+1,0}$ and $\zeta_X^{k,l}$ with $l > a$ factor through $\zeta_X^{0,a+1}$. According to Proposition 6, the cup product $\zeta_X^{a+1,0} \equiv 0$ vanishes identically. Hodge duality on the compact Kähler manifold X provides $\zeta_X^{0,a+1} \equiv 0$. The vanishing of $\zeta_X^{k,0}$ for $k > a$ implies the nonexistence of strict k-wedges $U_k \subset H^{1,0}(X)$ (cf. Lemma 7 *(ii)*). Applying Corollary 9, one concludes that $g_k = 0$ for all $k > a$, Q.E.D.

Lemma 13 *Let X be a compact Kähler manifold with irregularity $h^{1,0} > 0$, Albanese dimension $a > 0$ and Albanese genera g_k, $1 \leq k \leq a$. Then the ranks of the cup products*

$$\zeta_X^{i,j} : \wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X) \to H^{i+j}(X, \mathbb{C})$$

satisfy the following lower bounds

(i) $rk\zeta_X^{i,j} \geq \binom{a}{i+j}$ *for* $i, j \in \mathbb{N} \cup \{0\}$, $2 \leq i+j \leq a$;

(ii) $rk\zeta_X^{i,j} \geq \binom{g_k - i}{j}$, $rk\zeta_X^{j,i} \geq \binom{g_k - i}{j}$ *if* $g_k > 0$ *for some* $0 \leq i \leq j$, $2 \leq i+j \leq k \leq a$;

(iii) $rk\zeta_X^{i,j} \geq (i+j)(h^{1,0} - i - j) + 1$ *if* $g_k = 0$ *for all* $1 \leq k < i+j \leq a$;

(iv) $rk\zeta_X^{1,1} \geq 2a - 1$.

Proof: *(i)* By Proposition 6 one has $\zeta_X^{a,0} \not\equiv 0$. Let $\omega_1, \ldots, \omega_a \in H^{1,0}(X)$ be global holomorphic forms with $\varphi := \zeta_X^{a,0}(\omega_1 \wedge \ldots \wedge \omega_a) \neq 0$. Consider the subspace

$$T^{i,j} := Span_{\mathbb{C}}(\omega_{t_1} \wedge \ldots \wedge \omega_{t_i} \otimes \overline{\omega_{s_1}} \wedge \ldots \wedge \overline{\omega_{s_j}} \ \mid \ 1 \leq t_1 < \ldots < t_j < s_1 < \ldots < s_j \leq a)$$

of $\wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X)$. We claim that $T^{i,j}$ is injected by $\zeta_X^{i,j}$, so that $rk\zeta_X^{i,j} \geq \dim_{\mathbb{C}} \zeta_X^{i,j}(T^{i,j}) = \dim_{\mathbb{C}} T^{i,j} = \binom{a}{i+j}$. Let us observe that $\varphi \in H^{a,0}(X)$ is primitive, i.e., $\varphi \wedge \Omega^{n+1-a} \in H^{n+1,n+1-a}(X) = 0$ where Ω stands for the Kähler form of X and $n = \dim_{\mathbb{C}} X$. By the degeneracy of the Hodge Hermitian form on the primitive cohomologies, one has $(-1)^{\frac{a(a-1)}{2}} \left(\sqrt{-1}\right)^a \int_X \varphi \wedge \overline{\varphi} \wedge \Omega^{n-a} > 0$ and the cohomology class $\zeta_X^{a,a}(\varphi \wedge \overline{\varphi}) \neq 0$ in $H^{a,a}(X)$. For any $\alpha = \sum_{t,s} c_{t,s} \omega_{t_1} \wedge \ldots \wedge \omega_{t_i} \otimes \overline{\omega_{s_1}} \wedge \ldots \wedge \overline{\omega_{s_j}} \in Ker\zeta_X^{i,j} \cap T^{i,j}$, $c_{t,s} \in \mathbb{C}$, let us wedge $\zeta_X^{i,j}(\alpha) = d\beta_1$ by $\left(\wedge_{k \in \{1,\ldots,a\} \setminus \{t_1,\ldots,t_i\}} \omega_k\right) \wedge \left(\wedge_{l \in \{1,\ldots,a\} \setminus \{s_1,\ldots,s_j\}} \overline{\omega_l}\right)$, to obtain $\pm c_{t,s} \varphi \wedge \overline{\varphi} = d\beta_2$ for appropriate differential forms β_1, β_2. That implies the vanishing of all the complex coefficients $c_{t,s}$ of α and justifies that $Ker\zeta_X^{i,j} \cap T^{i,j} = 0$, whereas $\zeta_X^{i,j}(T^{i,j}) \simeq T^{i,j}$.

(ii) If $g_k > 0$ for some $i+j \leq k \leq a$, then according to Corollary 9 there is a strict k-wedge $U_k \subset H^{1,0}(X)$ of $\dim_{\mathbb{C}} U_k = g_k \geq k+1$. Let $u_1, \ldots, u_{g_k}$ be a $\mathbb{C}$-basis of U_k and

$$A_k^{i,j} := u_1 \wedge \ldots \wedge u_i \otimes_{\mathbb{C}} \wedge^j Span_{\mathbb{C}}\left(\overline{u_{i+1}}, \ldots, \overline{u_{g_k}}\right), \quad B_k^{i,j} := \wedge^i Span_{\mathbb{C}}\left(u_{j+1}, \ldots, u_{g_k}\right) \otimes_{\mathbb{C}} \overline{u_1} \wedge \ldots \wedge \overline{u_j}$$

be subspaces of $\wedge^i H^{1,0}(X) \otimes_{\mathbb{C}} \wedge^j H^{0,1}(X)$. We claim that $Ker\zeta_X^{i,j} \cap A_k^{i,j} = 0$ and $Ker\zeta_X^{i,j} \cap B_k^{i,j} = 0$, so that

$$rk\zeta_X^{i,j} \geq \max\left(\dim_{\mathbb{C}} \zeta_X^{i,j}(A_k^{i,j}), \dim_{\mathbb{C}} \zeta_X^{i,j}(B_k^{i,j})\right) = \max\left(\dim_{\mathbb{C}} A_k^{i,j}, \dim_{\mathbb{C}} B_k^{i,j}\right) =$$

$$\max\left(\binom{g_k - i}{j}, \binom{g_k - j}{i}\right) = \binom{g_k - \min(i,j)}{\max(i,j)},$$

as far as $\binom{g_k - i}{j} : \binom{g_k - j}{i} = \prod_{s=1}^{j-i} \left(\frac{g_k - j + s}{i+s}\right) > 1$ for $i < j$ and $g_k > i+j$. Let us assume that $\psi = u_1 \wedge \ldots \wedge u_i \otimes_{\mathbb{C}} \left(\sum_{i+1 \leq s_1 < \ldots < s_j \leq g_k} c_s \overline{u_{s_1}} \wedge \ldots \wedge \overline{u_{s_j}}\right) \in Ker\zeta_X^{i,j} \cap A_k^{i,j}$ for some $c_s \in \mathbb{C}$. If Ω

is the Kähler form of X and $n=\dim_{\mathbb{C}} X$ then $0=\int_X \psi\wedge\overline{\psi}\wedge\Omega^{n-i-j}=\pm\int_X\varphi\wedge\overline{\varphi}\wedge\Omega^{n-i-j}$ for $\varphi:=u_1\wedge\ldots\wedge u_i\wedge\left(\sum_{i+1\leq s_1<\ldots<s_j\leq g_k} c_s u_{s_1}\wedge\ldots\wedge u_{s_j}\right)\in\wedge^{i+j}U_k$. As far as $\zeta_X^{n+1,n+1-i-j}\left(\varphi\wedge\Omega^{n+1-i-j}\right)\in H^{n+1,n+1-i-j}(X)=0$, the form φ is primitive and $0=\int_X\varphi\wedge\overline{\varphi}\wedge\Omega^{n-i-j}=\int_X\zeta_X^{i+j,0}(\varphi)\wedge\overline{\zeta_X^{i+j,0}(\varphi)}\wedge\Omega^{n-i-j}$ implies that $\zeta_X^{i+j,0}(\varphi)=0$, according to the nondegeneracy of the Hodge Hermitian form on the primitive φ. In other words, $\varphi\in Ker\zeta_X^{i+j,0}\cap\left(\wedge^{i+j}U_k\right)$. However, $Ker\zeta_X^{k,0}\cap\left(\wedge^k U_k\right)=0$ by the strictness of the k-wedge U_k. The factorization of $\zeta_X^{k,0}$ through $\zeta_X^{i+j,0}$ for $i+j\leq k$ implies that $Ker\zeta_X^{i+j,0}\cap\left(\wedge^{i+j}U_k\right)=0$, whereas $\varphi=0$. Due to the $\mathbb{C}$-linear independence of $u_1\wedge\ldots\wedge u_i\wedge u_{s_1}\wedge\ldots\wedge u_{s_j}$ with $i+1\leq s_1<\ldots<s_j\leq g_k$ there follow $c_s=0$ for all $s=(s_1,\ldots,s_j)$. Consequently, $Ker\zeta_X^{i,j}\cap A_k^{i,j}=0$. Similar considerations justify $Ker\zeta_X^{i,j}\cap B_k^{i,j}=0$.

(iii) Let us consider the punctured cone

$$\mathcal{C}_X^{i,j}:=\{\omega_1\wedge\ldots\wedge\omega_i\wedge\overline{\omega_{i+1}}\wedge\ldots\wedge\overline{\omega_{i+j}}\ \mid\ \omega_1,\ldots,\omega_{i+j}\in H^{1,0}(X),\omega_1\wedge\ldots\wedge\omega_i\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{i+j}\neq 0\}.$$

There is a real diffeomorphism $\mathcal{C}_X^{i,j}\to\mathcal{C}_X^{i+j,0}:=\{\omega_1\wedge\ldots\wedge\omega_i\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{i+j}\neq 0|\omega_1,\ldots,\omega_{i+j}\in H^{1,0}(X)\}$ onto the cone of the decomposable elements of $\wedge^{i+j}H^{1,0}(X)$ with punctured origin. The projectivization of $\mathcal{C}_X^{i+j,0}$ is the Grassmannian manifold $Grass(i+j,H^{1,0}(X))$, so that $\dim_{\mathbb{C}}\mathcal{C}_X^{i+j,0}=(i+j)(h^{1,0}-i-j)+1$. We claim that $Ker\zeta_X^{i,j}\cap\mathcal{C}_X^{i,j}=\emptyset$, in order to estimate

$$rk\zeta_X^{i,j}=\left(\begin{array}{c}h^{1,0}\\ i\end{array}\right)\left(\begin{array}{c}h^{1,0}\\ j\end{array}\right)-\dim_{\mathbb{C}}Ker\zeta_X^{i,j}\geq\dim_{\mathbb{C}}\mathcal{C}_X^{i,j}=\dim_{\mathbb{C}}\mathcal{C}_X^{i+j,0}.$$

Let us assume the opposite,i.e., $\psi=\omega_1\wedge\ldots\wedge\omega_i\wedge\overline{\omega_{i+1}}\wedge\ldots\wedge\overline{\omega_{i+j}}\in Ker\zeta_X^{i,j}\cap\mathcal{C}_X^{i,j}$. Then $0=\int_X\psi\wedge\overline{\psi}\wedge\Omega^{n-i-j}=\pm\int_X\varphi\wedge\overline{\varphi}\wedge\Omega^{n-i-j}$ for $\varphi=\omega_1\wedge\ldots\wedge\omega_i\wedge\omega_{i+1}\wedge\ldots\wedge\omega_{i+j}$. On a compact Kähler manifold X of $\dim_{\mathbb{C}}X=n$, the fact that $\zeta_X^{n+1,n-i-j+1}(\varphi\wedge\Omega^{n-i-j+1})\in H^{n+1,n-i-j+1}(X)$ reveals the primitiveness of φ. Then the vanishing of $\int_X\varphi\wedge\overline{\varphi}\wedge\Omega^{n-i-j}$ implies $\zeta_X^{i+j,0}(\varphi)=0$. Let V_φ be the $\mathbb{C}$-span of $\omega_1,\ldots,\omega_{i+j}$. By a decreasing induction on $1\leq k\leq i+j$ will be checked that $\wedge^k V_\varphi\subset Ker\zeta_X^{k,0}$. We have already seen that $\mathbb{C}\varphi=\wedge^{i+j}V_\varphi\subset Ker\zeta_X^{i+j,0}$. For any natural number $k<i+j$ and a decomposable form $0\neq\psi':=\omega_{s_1}\wedge\ldots\wedge\omega_{s_k}\in\wedge^k V_\varphi$, $\omega_{s_i}\in V_\varphi$, there exists $\omega_{s_{k+1}}\in V_\varphi$ with $\psi:=\psi'\wedge\omega_{s_{k+1}}\neq 0$. By the inductional hypothesis $\psi\in\wedge^{k+1}V_\varphi\subset Ker\zeta_X^{k+1,0}$. If $\zeta_X^{k,0}(\psi')\neq 0$ then for $V_\psi:=Span_{\mathbb{C}}(\omega_{s_1},\ldots,\omega_{s_k},\omega_{s_{k+1}})$ there hold $\zeta_X^{k,0}(\wedge^k V_\psi)\neq 0$ and $\zeta_X^{k+1,0}(\wedge^{k+1}V_\psi)=0$. In other words, $V_\psi\subset H^{1,0}(X)$ appears to be a k-wedge and according to Lemma 10, there is a strict l-wedge $U_l\subset V_\psi$ for some $1\leq l\leq k$. However, $g_l\geq\dim_{\mathbb{C}}U_l>0$ for $l<i+j$ contradicts the assumptions of *(iii)*. Consequently, $\zeta_X^{k,0}(\psi')=0$ for all decomposable elements of $\wedge^k V_\varphi$, whereas $\wedge^k V_\varphi\subset Ker\zeta_X^{k,0}$ for all $1\leq k\leq i+j$. In particular, $V_\varphi\subset Ker\zeta_X^{1,0}=KerId_{H^{1,0}(X)}$ is an absurd, justifying $Ker\zeta_X^{i,j}\cap\mathcal{C}_X^{i,j}=\emptyset$. Let us observe that this estimate generalizes Amorós' Lemma 5 *(i)*.

(iv) According to Proposition 6, there exist $\omega_1,\ldots,\omega_a\in H^{1,0}(X)$ with $0\neq\omega_1\wedge\ldots\wedge\omega_a\wedge\overline{\omega_1}\wedge\ldots\wedge\overline{\omega_a}\in H^{a,a}(X)$. For $a=1$ it is immediate that $rk\zeta_X^{1,1}\geq 1$. For $a\geq 2$ we assert that the subspace

$$W:=Span_{\mathbb{C}}(\omega_1\otimes_{\mathbb{C}}\overline{\omega_1},\ \ \omega_1\otimes_{\mathbb{C}}\overline{\omega_i},\ \ \omega_i\otimes_{\mathbb{C}}\overline{\omega_1}\ \mid\ 2\leq i\leq a)$$

of $H^{1,0}(X)\otimes_{\mathbb{C}}H^{0,1}(X)$ is embedded in $H^{1,1}(X)$ by the cup product $\zeta_X^{1,1}$. Indeed, if $\alpha=b_0\omega_1\wedge\overline{\omega_1}+\sum_{i=2}^a b_i\omega_1\wedge\overline{\omega_i}+\sum_{i=2}^a c_i\omega_i\wedge\overline{\omega_1}=d\beta$ for some $b_0,b_i,c_i\in\mathbb{C}$ and a 1-form β, then $\alpha\wedge\overline{\alpha}=d(\beta\wedge d\overline{\beta})=\sum_{i=2}^a\sum_{j=2}^a(\overline{b_i}b_j+c_i\overline{c_j})\omega_1\wedge\overline{\omega_1}\wedge\omega_i\wedge\overline{\omega_j}$. Introducing $\sigma_i:=\omega_2\wedge\ldots\wedge\omega_{i-1}\wedge\omega_{i+1}\wedge\ldots\wedge\omega_a$ for $2\leq i\leq a$, one obtains the vanishing of $\alpha\wedge\overline{\alpha}\wedge\sigma_i\wedge\overline{\sigma_i}=d(\beta\wedge d\overline{\beta}\wedge\sigma_i\wedge\overline{\sigma_i})=\pm(|b_i|^2+|c_i|^2)\omega_1\wedge\ldots\wedge\omega_a\wedge\overline{\omega_1}\wedge\ldots\wedge\overline{\omega_a}\in H^{a,a}(X)$. By the choice of $\omega_1,\ldots,\omega_a$, there follow $b_i=0$ and $c_i=0$ for all $2\leq i\leq a$, whereas $\alpha=b_0\omega_1\wedge\overline{\omega_1}=d\beta$. The assumption $b_0\neq 0$ would imply $\omega_1\wedge\ldots\omega_a\wedge\overline{\omega_1}\wedge\ldots\wedge\overline{\omega_a}=\frac{(-1)^{a-1}}{b_0}d\beta\wedge\omega_2\wedge\ldots\wedge\omega_a\wedge\overline{\omega_2}\wedge\ldots\wedge\overline{\omega_a}=d\left(\frac{(-1)^{a-1}\beta}{b_0}\wedge\omega_2\wedge\ldots\omega_a\wedge\overline{\omega_2}\wedge\ldots\wedge\overline{\omega_a}\right)=0\in H^{a,a}(X)$, which is an absurd. Therefore $b_0=0$ and $Ker\zeta_X^{1,1}\cap W=0$, whereas $rk\zeta_X^{1,1}\geq\dim_{\mathbb{C}}\zeta_X^{1,1}(W)=\dim_{\mathbb{C}}W=2a-1$.

As far as $\wedge^{h^{1,0}+1}H^{1,0}(X)=0$, the Albanese dimension $a\leq h^{1,0}$. Thus, in the case of $g_1=0$ Amorós' lower bound $rk\zeta_X^{1,1}\geq 2h^{1,0}-1$ from Lemma 5 *(ii)* is better than $rk\zeta_X^{1,1}\geq 2a-1$, Q.E.D.

6 Proofs of the main results

The Betti number $b_2(G) := \dim_{\mathbb{C}} H^2(G,\mathbb{C}) = rkH_2(G,\mathbb{Z})$ of an arbitrary finitely presented group G can be expressed by the means of the following

Theorem 14 (Hopf [8], [4], [6]) *Let $F = \langle x_1,\dots,x_s\rangle$ be a free group, R be the normal subgroup of F, generated by $y_1,\dots,y_r \in F$ and $G = F/R$. Then there is an exact sequence of group homologies*

$$0 \to H_2(G,\mathbb{Z}) \to H_1(R,\mathbb{Z})_G \to H_1(F,\mathbb{Z}) \to H_1(G,\mathbb{Z}) \to 0$$

where the subscript G stands for the G-coinvariants of the adjoint action.

The ranks of the aforementioned homology groups, will be calculated by the means of the following

Lemma 15 *Let $F = \langle x_1,\dots,x_s\rangle$ be a free group, R_o be the subgroup of F, generated by $y_1,\dots y_r \in F$, R be the normal subgroup of F, generated by R_o and $G = F/R$. Then*

(i) the coinvariants $(abR)_G = (abR)_F = R/[F,R]$ are isomorphic to the image $R_o/(R_o \cap [F,R])$ of the F-coinvariants epimorphism $abR_o \to abR_o/K$ with kernel $K := (R_o \cap [F,R])/[R_o,R_o]$. In particular, $(abR)_G$ is a finitely generated abelian group of $rk(abR)_G = r-k$ where $k := rkK$.

(ii) $s = rk(abF) \geq rk(ab(G))$ with equality exactly for $R \subset [F,F]$.

Proof: Let us recall from [10] the isomorphism $H_1(\Gamma,\mathbb{Z}) \simeq ab\Gamma := \Gamma/[\Gamma,\Gamma]$ for an arbitrary group Γ. The adjoint action of F on its normal subgroup R descends to an adjoint action on $abR = R/[R,R]$, as far as $[R,R]$ is also normal in F. Since the adjoint action of R centralizes abR, the F-action on abR coincides with the G-action, as well as the corresponding coinvariants $(abR)_F = (abR)_G$. The kernel of the coinvariants epimorphism $abR \to (abR)_F$ is generated by $frf^{-1}r^{-1}[R,R]$ for $f \in F$, $r \in R$. Therefore $(abR)_F = \frac{abR}{[F,R]/[R,R]} = R/[F,R]$.

The normal subgroup R of F is generated by $f^{-1}y_jf$ for $1 \leq j \leq r$, $f \in F$. Therefore $[F,R]$ is generated by $(f_1^{-1}y_j^{-1}f_1)(f_2^{-1}y_jf_2)$ for $1 \leq j \leq r$, $f_1,f_2 \in F$. In particular, $y_j^{-1}f^{-1}y_jf \in [F,R]$, whereas $f^{-1}y_jf[F,R] = y_j[F,R]$ for all $1 \leq j \leq r$ and $f \in F$, so that any coset $r[F,R] \in R/[F,R]$ has a representative $r_o \in R_o$, $r_o[F,R] = r[F,R]$. That is why, the natural map $\psi : R_o \to R/[F,R]$, $\psi(r_o) = r_o[F,R]$ is an epimorphism with $Ker\psi = R_o \cap [F,R]$. Thus, $R_o/(R_o \cap [F,R])$ is isomorphic to $R/[F,R]$. Representing $R_o/(R_o \cap [F,R]) = (abR_o)/K$ by $K := (R_o \cap [F,R])/[R_o,R_o]$, one concludes that $R/[F,R]$ is a finitely generated abelian group. Clearly, $rk\,(R/[F,R]) = rk(abR_o) - rkK = r-k$.

(ii) The abelianization is a right exact functor, so that the epimorphism $\alpha : F \to G$ induces an epimorphism $\beta : abF \simeq \mathbb{Z}^s \to abG$. In particular, $s = rk(abF) \geq rk(ab(G))$. If $s = rk(ab(G))$ then β has to be an isomorphism. On one hand, $F \to abF \simeq abG$ has kernel $[F,F]$. On the other hand, the composed map $F \to G \to abG$ contains R in its kernel, so that $R \subset [F,F]$. Conversely, if $R \subset [F,F]$ then $abG = \frac{G}{[F,F]/R} \simeq abF$, Q.E.D.

Proof of Proposition 1: Hopf's Theorem 14 and Lemma 15 imply that $b_2(\pi_1(X)) = r-k-s+2h^{1,0}$, as far as $b_1(\pi_1(X)) = b_1(X) = 2h^{1,0}$. On the other hand, Lemma 11 provides $b_2(\pi_1(X)) \geq 2rk\zeta_X^{2,0} + rk\zeta_X^{1,1}$. The proof is completed by the following immediate consequence of Lemma 13 and Lemam 5*(ii)*:

Corollary 16 *Let X be a compact Kähler manifold with irregularity $h^{1,0} > 0$, Albanese dimension $a > 0$ and Albanese genera g_k, $1 \leq k \leq a$. Then the Betti numbers $b_2 = b_2(\pi_1(X))$ and $b_2 = b_2(X) = b_{2n-2}(X)$ are subject to the following lower bounds:*

(i) $b_2 \geq 1$ for $a = 1$;

(ii) $b_2 \geq \max\left(a(a-1), g_k(g_k-1) \ \mid \ 2 \leq k \leq a\right) + \max\left(\frac{a(a-1)}{2}, 2a-1, g_k-1 \ \mid \ 2 \leq k \leq a\right)$ for $h^{1,0} \geq g_1 \geq 2$, $a \geq 2$;

*(iii) $b_2 \geq \max\left(4h^{1,0}-6, a(a-1), g_k(g_k-1) \ \mid \ 2 \leq k \leq a\right) +$
$\max\left(2h^{1,0}-1, \frac{a(a-1)}{2}, g_k-1 \ \mid \ 2 \leq k \leq a\right)$ for $h^{1,0} \geq 2$, $a \geq 2$, $g_1 = 0$.*

Proof of Proposition 2: First of all, let us observe that $Im\zeta_*^{j,m-j} \cap \left(\sum_{s\neq j} Im\zeta_*^{s,m-s} \right) = 0$ for either of the cup products $\zeta_{\pi_1(X)}^{j,m-j}$ or $\zeta_X^{j,m-j}$. Therefore $b_m(*) \geq \sum_{j=0}^m rk\zeta_*^{j,m-j}$. Lemma 11 has established that $rk\zeta_{\pi_1(X)}^{j,m-j} \geq rk\zeta_X^{j,m-j}$. The lower bounds on $rk\zeta_X^{i,j}$ from Lemma 13 are invariant under a permutation of i with j. Combining them, one obtains $rk\zeta_X^{i,j} \geq \mu^{i,j}$ for $\mu^{i,j}$, defined in the statement of Proposition 2, $0 \leq i \leq j$, $3 \leq i+j \leq a$. Therefore, the Betti numbers of the compact Kähler manifold X and its fundamental group $\pi_1(X)$ are subject to the inequalities $b_{2i}(*) \geq 2\sum_{j=0}^{i-1} \mu^{j,2i-j} + \mu^{i,i}$, $b_{2i+1}(*) \geq 2\sum_{j=0}^{i} \mu^{j,2i+1-j}$ for $3 \leq 2i, 2i+1 \leq a$. In the case of $2n-a \leq m \leq 2n-3$, by Serre duality on the cohomologies of X there hold

$$h^{j,m-j}(X) = h^{n-j,n-m+j}(X) \geq rk\zeta_X^{n-j,n-m+j} \geq \mu^{n-\max(j,m-j),n-\min(j,n-j)}$$

as far as $3 \leq 2n-m \leq a$. Combining with Hodge duality $h^{n-j,j}(X) = h^{j,n-j}(X)$ for compact Kähler manifolds X, one justifies the last two announced inequalities, Q.E.D.

At first glance, Proposition 1 can be reformulated entirely in terms of the cohomologies of $\pi_1(X)$. However, there are several obstacles for doing that. First of all, any isomorphism $c^{(1)} : H^1(\pi_1(X), \mathbb{C}) \to H^1(X, \mathbb{C})$ allows to introduce $H^{i,j}(\pi_1(X)) := (c^{(1)})^{-1} H^{i,j}(X)$ for $(i,j) = (1,0)$ or $(0,1)$ and to endow $H^1(\pi_1(X), \mathbb{C}) = H^1(\pi_1(X), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ with a polarized Hodge structure. For an abstract finitely presented group G with $H^1(G, \mathbb{Z})$ of even rank $2q$, the polarized Hodge structures on $H^1(G, \mathbb{C}) = H^1(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$ are parametrized by the Siegel upper half-space $\mathcal{S} = Sp(q, \mathbb{R})/U_q$. Unfortunately, the cup products $\zeta_G^i : \wedge^i H^1(G, \mathbb{C}) \to H^i(G, \mathbb{C})$ are not invariant under the action of the symplectic group $Sp(q, \mathbb{R})$ and the counterparts of Albanese dimension $a = \max\{m \in \mathbb{N} \cup \{0\} \ | \ \zeta^{m,0}(\wedge^m H^{1,0}) \neq 0, \zeta^{m+1,0}(\wedge^{m+1} H^{1,0}) = 0\}$ and Albanese genera $g_k = \max\{g \in \mathbb{N} \cup \{0\} | \exists \text{ subspace } U \subset H^{1,0}, \dim_{\mathbb{C}} U = g, Ker[\zeta^k : \wedge^k U \to H^k] = 0, Im[\zeta^{k+1} : \wedge^{k+1} U \to H^{k+1}] = 0\}$ depend on $\xi \in \mathcal{S}$ (cf. Proposition 6 and Corollary 9). The corresponding notions cannot be defined in terms of real cup products. Namely, for the real point set $U^{\mathbb{R}} := Span_{\mathbb{R}}(u + \overline{u}, \sqrt{-1}u - \sqrt{-1}\overline{u} \ | \ u \in U)$ of $U \subset H^{1,0}$, the condition $\zeta^{2k+1}(\wedge^{2k+1} U^{\mathbb{R}}) = 0$ is necessary but not sufficient for $\zeta^{k+1}(\wedge^{k+1} U) = 0$. Finally, the cup product in de Rham cohomologies $H^*(X, \mathbb{C})$ of a compact Kähler manifold X with $\pi_1(X) = G$ are quotients of the corresponding cup products in $H^*(\pi_1(X), \mathbb{C})$. Thus, $\zeta_X^i(\wedge^i H^{1,0}(X)) = 0$ does not imply $\zeta_{\pi_1(X)}^i(\wedge^i H^{1,0}(\pi_1(X))_s) = 0$, $\xi \in \mathcal{S}$.

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